1. CLASSES OF GROUPS

§1.1. Definition and Examples

Having discussed the many facets of group theory from Volume 1, I would like to introduce a very useful system of notation which facilitates the further discussion of the subject.

A **class of groups** is a collection of groups $\mathfrak{X}$ such that:

(1) $\mathbf{1} \in \mathfrak{X}$ and

(1) If $G \cong H$ then $G \in \mathfrak{X}$ if and only if $H \in \mathfrak{X}$.

We call them 'classes' rather than 'sets' to avoid the set-theoretic paradox, known as the Russell Paradox.

The word ‘class’ refers to the intuitive concept of a collection of things. But we don’t allow classes to be elements of other classes.

When building up set theory rigorously we have to be very careful what we allow to be called a set. The Russell Paradox arises if we tried to contemplate the set $y = \{x \mid x \notin x\}$ because we would be forced to conclude that $y \in y$ if and only if $y \notin y$. When developing axiomatic set theory we consider sets as undefined objects with an undefined relation $x \in y$ which may or may not hold between sets x and y.

But here we don’t have to tread carefully because our classes are not assumed to be sets, even though their elements are sets.

As with subsets, we use the same notation, $\subseteq$ and $\subset$ for subclasses and proper subclasses. So $\mathcal{C} \subset \mathcal{A}$ is a compact way of saying that every cyclic group is abelian but not every abelian subgroup is cyclic.

Here **1** is the trivial group. Axiom (2) captures the point of view that we consider isomorphic groups to be essentially the same group, unless of course if one is a subgroup of the other. So we talk about *the* dihedral group of order 8 even though we can build a dihedral group of order 8 upon any set of 8 elements.

Examples 1: We’ll consider the following classes:
$\mathcal{I}$ = the class containing just the trivial group.
$\mathcal{F}$ = the class of finite groups;

$\mathcal{C}$ = the class of cyclic groups;
$\mathcal{A}$ = the class of abelian groups;
$\mathcal{D}$ = the class of all cyclic or dihedral groups (either finite or infinite);
$\mathcal{P}$ = the class of all *p*-groups;
Σ = the class of all finite symmetric groups.

We could consider the class $\mathcal{G}$ of all groups, but that would not be very useful. Nor do we consider the class of all permutation groups because that would be the same as $\mathcal{G}$ because every group is isomorphic to a group of permutations.

We define a group to be an **$\mathcal{X}$-group** if $G \in \mathcal{X}$.

§1.2. Products of Classes

If $\mathcal{X}$ and $\mathcal{Y}$ are group classes we define
$\mathcal{X}\mathcal{Y}$ = the class of all groups G such that G has a normal subgroup $H \in \mathcal{X}$ with $G/H \in \mathcal{Y}$.

Example 2: $\mathcal{D} \subset \mathcal{C}\mathcal{C}$ because every cyclic or dihedral group G has a normal subgroup H where H is cyclic and G/H is cyclic of order 2. But $\mathbf{C}_3 \times \mathbf{C}_3 \in \mathcal{C}\mathcal{C}$ yet isn't cyclic or dihedral.

The multiplication is not quite associative in that

$\mathcal{X}(\mathcal{Y}\mathcal{Z}) \subseteq (\mathcal{X}\mathcal{Y})\mathcal{Z}$ for all classes $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, though equality doesn't always hold.

Theorem 1: For all group classes $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$,

$$\mathcal{X}(\mathcal{Y}\mathcal{Z}) \subseteq (\mathcal{X}\mathcal{Y})\mathcal{Z}.$$

Proof: Let $G \in \mathcal{X}(\mathcal{Y}\mathcal{Z})$. Then G has a normal subgroup
$H \in \mathcal{X}$ such that $G/H \in \mathcal{Y}\mathcal{Z}$.
Then G/H has a normal subgroup $K/H \in \mathcal{Y}$ such that
$(G/H)/(K/H) \in \mathcal{Z}$.
By the 3rd Isomorphism Theorem,
$(G/H)/(K/H) \cong G/K$ and so $G/K \in \mathcal{Z}$.
Now K is a normal subgroup of H and so $K \in \mathcal{X}\mathcal{Y}$.
Hence $G \in (\mathcal{X}\mathcal{Y})\mathcal{Z}$. ✋☺

Example 3: $\mathbf{V}_4 \le \mathbf{A}_4 \le \mathbf{S}_4$ so $\mathbf{S}_4 \in (\mathcal{C}\mathcal{C})\mathcal{C}$.
But $\mathbf{S}_4 \notin \mathcal{C}(\mathcal{C}\mathcal{C})$ because it has no non-trivial cyclic normal subgroup. Hence $\mathcal{C}(\mathcal{C}\mathcal{C})$ is a proper subclass of $(\mathcal{C}\mathcal{C})\mathcal{C}$.

The trivial class, {**1**}, acts as the identity for this product. The product of group classes is also not commutative.

Example 4: $\mathbf{S}_4 \in \mathcal{A}\mathcal{C}$ but $\mathbf{S}_4 \notin \mathcal{C}\mathcal{A}$.

If $\mathcal{X}$ is any group class the we define

$$\mathcal{X}^0 = \mathcal{I},\ \mathcal{X}^{n+1} = \mathcal{X}^n\mathcal{X}.$$

Clearly $\mathcal{X}^1 = \mathcal{X}$ for all group classes.

Example 5: $\mathfrak{F}^n = \mathfrak{F}$ for all $n \geq 1$.

§1.3. Subnormal Series and Subgroups

If $H \leq G$ a **subnormal series** is a series of subgroups

$$G_0 = H < G_1 < \ldots < G_n.$$

such that each $G_i \lhd G_{i+1}$. The **quotients** of such a subnormal series are the non-trivial quotients G_{i+1}/G_i and its l**ength** is n. Since the relation of being a normal subgroup isn't transitive the subgroups are normal in the next but need not be normal in the whole group.

A subgroup H of a group G is defined to be **subnormal** if there exists a subnormal series from H to G. In such a case we write $H \lhd \lhd G$.

Example 6: The series

$$\langle(12)(34)\rangle < \mathbf{V}_4 < \mathbf{A}_4 < \mathbf{S}_4$$

is a subnormal series of length 3, whose quotients are isomorphic to $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_2$ respectively. Hence $\langle(12)(34)\rangle$ is a subnormal subgroup of $\mathbf{S}_4$.

Example 7: The subgroup $H = \langle(12)\rangle$ is not a subnormal subgroup of $\mathbf{S}_5$. This is because the only normal subgroups of $\mathbf{S}_5$ are 1, $\mathbf{A}_5$ and $\mathbf{S}_5$ and $\mathbf{A}_5$ is simple.

The only subnormal series that are possible for $\mathbf{S}_5$ are: $1 < \mathbf{A}_5 < \mathbf{S}_5$ and $1 < \mathbf{S}_5$. Neither passes through H.

If $\mathfrak{X}$ is a group class we define $\mathfrak{X}^\infty = \cup \mathfrak{X}^n$.

If G $\in \mathfrak{X}^\infty$ then G $\in \mathfrak{X}^n$ for some n and so there exists a subnormal series $G_0 = 1 < G_1 < \ldots < G_n$ for some n such that each $G_{i+1}/G_i \in \mathfrak{X}$. Clearly $\mathfrak{F}^\infty = \mathfrak{F}$.

Example 8: If $\mathfrak{X} = \{\mathbf{C}_p\}$ for some prime p then $\mathfrak{X}^\infty$ is the class of all finite p-groups. This is because the centre of a non-trivial finite p-group G is non-trivial.
If $1 < H \le Z(G)$ then $H \trianglelefteq G$ and G/H is a smaller p-group.

Clearly $(\mathfrak{X}^\infty)^\infty = \mathfrak{X}^\infty$ for any group class, $\mathfrak{X}$. I've left it as an exercise.

§1.4. Closure Operations

A **closure operation** is a function F that takes a group class $\mathfrak{X}$ to a group class F$\mathfrak{X}$ such that:
(1) F$\mathfrak{I} = \mathfrak{I}$;
(2) $\mathfrak{X} \subseteq$ F$\mathfrak{X}$;
(3) F$\mathfrak{X}$ = F$^2\mathfrak{X}$.

Examples 9:
S$\mathfrak{X}$ = the class of all subgroups of $\mathfrak{X}$-groups;
Q$\mathfrak{X}$ = the class of all quotients of $\mathfrak{X}$-groups;
P$\mathfrak{X}$ = the class of all groups G that have a normal $\mathfrak{X}$-subgroup with G/H $\in \mathfrak{X}$.

If F is a closure operation, a group class $\mathfrak{X}$ is **F-closed** if F$\mathfrak{X} = \mathfrak{X}$.

Examples 10: The following table shows which of these 6 group classes are closed under these 3 closure operations.

	$\mathcal{I}$	$\mathcal{F}$	$\mathcal{C}$	$\mathcal{A}$	$\mathcal{D}$	$\mathcal{P}$	Σ
S	√	√	√	√	√	√	×
Q	√	√	√	√	√	√	√
P	√	√	×	×	×	√	×

Most of these are obvious. Note that the only quotient groups of $\mathbf{S}_n$ are $\mathbf{S}_n$, $\mathbf{S}_n/\mathbf{A}_n \cong \mathbf{C}_2 \cong \mathbf{S}_2$ and $\mathbf{1} = \mathbf{S}_1$ and, in the case of $n = 4$, $\mathbf{S}_4/\mathbf{V}_4 \cong \mathbf{S}_3$.

Theorem 2:
(1) If $\mathfrak{X}$ and $\mathfrak{Y}$ are S-closed then so is $\mathfrak{X}\mathfrak{Y}$.
(2) If $\mathfrak{X}$ and Y are Q-closed then so is $\mathfrak{X}\mathfrak{Y}$.
Proof:
(1) Suppose that $\mathfrak{X}$, $\mathfrak{Y}$ are S-closed. We prove that $\mathfrak{X}\mathfrak{Y}$ is S-closed.

Let $G \in \mathfrak{X}\mathfrak{Y}$ and let $K \trianglelefteq G$ where $K \in \mathfrak{X}$ and $G/K \in \mathfrak{Y}$.

Since $HK/K \le G/K$, $HK/K \in \mathfrak{Y}$.

Then by the 2nd Isomorphism Theorem,

$H/H\cap K \cong HK/K \in \mathfrak{Y}$. Hence $H/H\cap K \in \mathfrak{Y}$.

Since $H\cap K \le K$, $H\cap K \in \mathfrak{X}$.

Thus $H \in \mathfrak{X}\mathfrak{Y}$.

(2) Suppose that $\mathfrak{X}$, $\mathfrak{Y}$ are Q-closed.

Let G $\in \mathfrak{X}\mathfrak{Y}$ and let K $\trianglelefteq$ G where K $\in \mathfrak{X}$ and G/K $\in \mathfrak{Y}$.

(2) Let G $\in \mathfrak{X}\mathfrak{Y}$ and let K $\leq$ G where K $\in \mathfrak{X}$ and G/K $\in \mathfrak{Y}$. Let H $\leq$ G.

By the 3rd Isomorphism Theorem

$$(G/H)/(HK/H) \cong G/HK \in (G/K)/(HK/K) \in \mathfrak{Y}.$$

By the 2nd Isomorphism Theorem

$$HK/H \cong K/H\cap K \in \mathfrak{X}.$$

Hence G/H $\in \mathfrak{X}\mathfrak{Y}$. ✋☺

Corollary:

(1) If $\mathfrak{X}$ is and $\mathfrak{Y}$ are S-closed then so is $\mathfrak{X}^n$ for all *n*.

(2) If $\mathfrak{X}$ is and $\mathfrak{Y}$ are Q-closed then so is $\mathfrak{X}^n$ for all *n*.

Theorem 3:

(1) If $\mathfrak{X}$ is S-closed then so is $\mathfrak{X}^\infty$.

(2) If $\mathfrak{X}$ is Q-closed then so is $\mathfrak{X}^\infty$.

(3) If $\mathfrak{X}$ is P-closed then so is $\mathfrak{X}^\infty$.

Proof: (1) and (2) follow immediately from the above corollary.

(3) Suppose $\mathfrak{X}$ is P-closed and suppose that H, G/H $\in \mathfrak{X}^\infty$. Then there exists L $\trianglelefteq$ G and K $\trianglelefteq$ H such that K, H/K, L/H and G/L, which is isomorphic to (G/H)/(L/H), all belong to $\mathfrak{X}^\infty$.

Suppose that K $\in \mathfrak{X}^r$, H/K $\in \mathfrak{X}^s$, L/H $\in \mathfrak{X}^u$ and G/L $\in \mathfrak{X}^v$. Then G $\in \mathfrak{X}^{r+s+u+v}$ and so G $\in \mathfrak{X}^\infty$. ✋☺

EXERCISES FOR CHAPTER 1

Exercise 1: For each of the following statements determine whether it is true or false.

(1) $S_3 \in \mathfrak{D}$.

(2) $\mathfrak{D} = \mathfrak{C}^2$.

(3) For all group classes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ we have $\mathfrak{X}(\mathfrak{Y}\mathfrak{Z}) = (\mathfrak{X}\mathfrak{Y})\mathfrak{Z}$.

(4) $\mathbf{S}_4 \in \mathfrak{A}\mathfrak{D}$.

(5) $\langle(12)\rangle$ is a subnormal subgroup of $\mathbf{S}_4$.

(6) $\mathfrak{L}$ is N-closed, where N$\mathfrak{X}$ is the class of all normal subgroups of $\mathfrak{X}$-groups and $\mathfrak{L}$ is the class of all finite alternating groups, A_n.

Exercise 2:

Let $\mathfrak{K} = \{\mathbf{S}_3, \mathbf{A}_4, \mathbf{S}_4\}$. Show that $\mathfrak{K} \cup \mathfrak{C}$ is Q-closed but not S-closed.

Exercise 3:

(a) Find all the subnormal subgroups of $\mathbf{S}_5$.

(b) Find the number of subnormal subgroups of $\mathbf{S}_4$. (Include S_4 itself).

Exercise 4:

Define products of closure operators by defining

$$AB\mathfrak{X} = A(B\mathfrak{X}).$$

(a) Prove that $S^2 = S$ and $Q^2 = Q$.

Exercise 5:

(a) Prove that for all $\mathfrak{X}$, $SQ\mathfrak{X} \subseteq QS\mathfrak{X}$.

(b) Find group G, H such that if $\mathfrak{X} = \{G\}$, $H \in QS\mathfrak{X}$ but $H \notin SQ\mathfrak{X}$.

SOLUTIONS FOR CHAPTER 1

Exercise 1:

(1) TRUE: $\mathbf{S}_3 \cong \mathbf{D}_6$.

(2) TRUE: $\mathbf{D}_{2n} = \langle A, B \mid A^{2n}, B^2, BA = A^{-1}B\rangle$ and

$\mathbf{D}_\infty = \langle A, B \mid B^2, BA = A^{-1}B\rangle$.

In each case $\langle A\rangle$ is a cyclic normal subgroup whose quotient is cyclic of order 2.

(3) FALSE

(4) TRUE: $\mathbf{V}_4$ is a normal subgroup of $\mathbf{S}_4$ whose quotient is isomorphic to $\mathbf{S}_3$ and hence to $\mathbf{D}_6$.

(5) FALSE

(6) FALSE: $\mathbf{V}_4$ is a normal subgroup of $\mathbf{A}_4$ but is not, itself, isomorphic to an alternating group.

Exercise 2: $\mathbf{V}_4 \leq \mathbf{S}_4$ but, while $\mathbf{S}_4 \in \mathcal{K}$, $\mathbf{V}_4 \notin \mathcal{K}$ so $\mathcal{K} \cup \mathcal{C}$ is not S-closed.

Quotients of cyclic group as cyclic.

The proper quotient groups of $\mathbf{S}_3$ and $\mathbf{A}_4$ are all cyclic.

The proper quotient groups of $\mathbf{S}_4$ are $\mathbf{C}_2$ and $\mathbf{S}_3$.

Exercise 3:

(a) The only non-trivial subnormal series for $\mathbf{S}_5$ are:

$$1 < \mathbf{S}_5 \text{ and } \mathbf{1} < \mathbf{A}_5 < \mathbf{S}_5$$

so the only subnormal subgroups are $\mathbf{1}$, $\mathbf{A}_5$ and $\mathbf{S}_5$.

(b) The subgroups of $\mathbf{S}_4$ are: 1, the cyclic subgroups, $\mathbf{V}_4$, $\mathbf{A}_4$, the subgroups isomorphic to $\mathbf{S}_3$ and $\mathbf{S}_4$.

Because $\mathbf{A}_4$, $\mathbf{V}_4$ and $\mathbf{1}$ are the only normal subgroups of $\mathbf{S}_4$ a subnormal series that ends in $\mathbf{S}_4$ must end in $\mathbf{1} < \mathbf{S}_4$, $\mathbf{V}_4 < \mathbf{S}_4$ or $\mathbf{A}_4 < \mathbf{S}_4$. So $\mathbf{1}$, $\mathbf{V}_4$, $\mathbf{A}_4$ and $\mathbf{S}_4$ are subnormal.

Now the only longer subnormal series are sections of the subnormal series $1 < \langle(\times\times)(\times\times)\rangle < \mathbf{V}_4 < \mathbf{A}_4 < \mathbf{S}_4$ for any one of the 3 cyclic subgroups of the form $\langle(\times\times)(\times\times)\rangle$. These 3 subgroups will be subnormal in S_4. But no other subgroups are possible and so altogether there are 7 such subnormal subgroups.

Exercise 4:

(a) $\mathbf{S^2 = S}$**:**

If $H \leq K \leq G$ then $H \leq G$ so $S^2\mathfrak{X} \subseteq S\mathfrak{X}$ for any $\mathfrak{X}$.

If $H \leq G$ then $H \leq G \leq G$ and so $H \in S^2\mathfrak{X}$.

Hence $S^2\mathfrak{X} \supseteq S\mathfrak{X}$ for any $\mathfrak{X}$.

$\mathbf{Q^2 = Q}$**:** A quotient group of the quotient G/K has the form (G/K)/(H/K) where $H, K \trianglelefteq G$ and $K \leq H$.

By the 3rd Isomorphism Theorem, this is isomorphic to G/H and so $Q^2\mathfrak{X} \subseteq Q\mathfrak{X}$ for any $\mathfrak{X}$.

Since $G/1 \cong G$ for any G, $G/H \cong (G/1)(H/1)$ and so

$$Q^2\mathfrak{X} \supseteq Q\mathfrak{X} \text{ for any } \mathfrak{X}.$$

Exercise 5:

(a) Let $A \in SQ\mathfrak{X}$. Then there exists a group $G \in \mathfrak{X}$, with a normal subgroup K, where S is isomorphic to a subgroup of G/K.

Now a subgroup of G/K has the form L/K where $K \le L \le G$. So there exists such an L where $L/K \cong A$. But this shows that $A \in QS\mathfrak{X}$. Hence $SQ\mathfrak{X} \subseteq QS\mathfrak{X}$.

(b) $\mathbf{S}_3 \cong \mathbf{A}_4/\mathbf{V}_4$ so $\mathbf{S}_3 \in QS\{\mathbf{A}_5\}$.

Apart from A_5/A_5, the only quotient of $\mathbf{A}_5$ is $\mathbf{A}_5$ itself. Yet $\mathbf{A}_5$ has no subgroup isomorphic to $\mathbf{S}_3$. Why not?

Suppose that $H \le \mathbf{A}_5$ and $H \cong \mathbf{S}_3$.

Now $\mathbf{S}_3 \cong \mathbf{D}_6 = \langle A, B \mid A^3, B^2, BA = A^{-1}B\rangle$.

The element $\alpha \in \mathbf{A}_5$ that corresponds to A must be a 3-cycle. Without loss of generality let it be (123). Now the permutation β, corresponding to B, has order 2, and it can't be a 2-cycle, so it must have cycle structure (××)(××).

Since $B^{-1}AB = A^{-1}$, $\beta^{-1}\alpha\beta = \alpha^{-1}$ and so β must permute {1, 2, 3} and so fix 4, a contradiction.